Using A Transition Matrix To Model Events

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A transition matrix can be used to model events such as default, bankruptcy and the loss of a major customer. The transition matrix has two states of the world, W_1 and W_2, which represent ***survival*** and ***no survival*** [ex. the borrower did not default (survival) or the borrower did default (no survival)]. Our matrix will look similar to the transition matrix below. The p_{rc} variables within that matrix refer to the conditional probability of starting out in the state denoted by the row heading and ending up in the state denoted by the column heading. These probabilities are conditional in that the probability of ending up in state W_1 or W_2 depends on the current state. The subscripts r and c refer to row and column, respectively.

Transition Matrix:

$$\begin{array}{ccc} & W_1 & W_2 \\ W_1 & p_{11} & p_{12} \\ W_2 & p_{21} & p_{22} \end{array}$$

Transition Probabilities:

p_{11} = probability of ending up in state W_1 in period $t+1$ given that I am in state W_1 in period t
p_{12} = probability of ending up in state W_2 in period $t+1$ given that I am in state W_1 in period t
p_{21} = probability of ending up in state W_1 in period $t+1$ given that I am in state W_2 in period t
p_{22} = probability of ending up in state W_2 in period $t+1$ given that I am in state W_2 in period t

Example - Default Transition Matrix

The matrix below is a borrower default transition matrix where W_1 = no default (i.e. survival) and W_2 = default (i.e. no survival). Note that W_2 is an absorbing state. Once the borrower defaults he cannot leave this state. If this is not the case then provisions must be made for transitions out of W_2. Note that the probabilities in each row sum to one.

$$\begin{array}{ccc} & W_1 & W_2 \\ W_1 & 1-p & p \\ W_2 & 0 & 1 \end{array}$$

The random variable S_t will be the state in which the borrower finds himself at the end of period t. We will assume that the transition matrix is stationary in that the conditional probabilities do not change over time. Given this assumptions, the conditional probabilities at any time t are...

$$\begin{aligned} P[S_t = W_1 | S_{t-1} = W_1] &= 1-p \\ P[S_t = W_1 | S_{t-1} = W_2] &= 0 \\ P[S_t = W_2 | S_{t-1} = W_1] &= p \\ P[S_t = W_2 | S_{t-1} = W_2] &= 1 \end{aligned}$$

Period Zero:

At $t=0$ the borrower has not yet defaulted. The unconditional probabilities at $t=0$ are...

$$\begin{aligned} P[S_0 = W_1] &= 1 \\ P[S_0 = W_2] &= 0 \end{aligned}$$

Note that the unconditional probabilities must always sum to one.

Period One:

The unconditional probability of no default at $t = 1$ is the conditional probability of no default at $t = 1$ given no default at $t = 0$ ***times*** the unconditional probability of no default at $t = 0$ ***plus*** the conditional probability of no default at $t = 1$ given default at $t = 0$ ***times*** the unconditional probability of default at $t = 0$.

$$\begin{aligned}
P[S_1 = W_1] &= \Big[P[S_1 = W_1|S_0 = W_1] \times P[S_0 = W_1]\Big] + \Big[P[S_1 = W_1|S_0 = W_2] \times P[S_0 = W_2]\Big] \\
&= \Big[(1-p) \times 1\Big] + \Big[0 \times 0\Big] \\
&= \Big[1-p\Big]
\end{aligned}$$

The unconditional probability of default at $t = 1$ is the conditional probability of default at $t = 1$ given no default at $t = 0$ ***times*** the unconditional probability of no default at $t = 0$ ***plus*** the conditional probability of default at $t = 1$ given default at $t = 0$ ***times*** the unconditional probability of default at $t = 0$.

$$\begin{aligned}
P[S_1 = W_2] &= \Big[P[S_1 = W_2|S_0 = W_1] \times P[S_0 = W_1]\Big] + \Big[P[S_1 = W_2|S_0 = W_2] \times P[S_0 = W_2]\Big] \\
&= \Big[p \times 1\Big] + \Big[1 \times 0\Big] \\
&= 1 - \Big[1-p\Big]
\end{aligned}$$

Period Two:

The unconditional probability of no default at $t = 2$ is the conditional probability of no default at $t = 2$ given no default at $t = 1$ ***times*** the unconditional probability of no default at $t = 1$ ***plus*** the conditional probability of no default at $t = 2$ given default at $t = 1$ ***times*** the unconditional probability of default at $t = 1$.

$$\begin{aligned}
P[S_2 = W_1] &= \Big[P[S_2 = W_1|S_1 = W_1] \times P[S_1 = W_1]\Big] + \Big[P[S_2 = W_1|S_1 = W_2] \times P[S_1 = W_2]\Big] \\
&= \Big[(1-p) \times (1-p)\Big] + \Big[0 \times (1-(1-p))\Big] \\
&= \Big[1-p\Big]^2
\end{aligned}$$

The unconditional probability of default at $t = 2$ is the conditional probability of default at $t = 2$ given no default at $t = 1$ ***times*** the unconditional probability of no default at $t = 1$ ***plus*** the conditional probability of default at $t = 2$ given default at $t = 0$ ***times*** the unconditional probability of default at $t = 1$.

$$\begin{aligned}
P[S_2 = W_2] &= \Big[P[S_2 = W_2|S_1 = W_1] \times P[S_1 = W_1]\Big] + \Big[P[S_2 = W_2|S_1 = W_2] \times P[S_1 = W_2]\Big] \\
&= \Big[p \times (1-p)\Big] + \Big[1 \times (1-(1-p))\Big] \\
&= 1 - \Big[1-p\Big]^2
\end{aligned}$$

Equations For Any Period:

By reviewing the unconditional probability equations in periods 1 and 2 the pattern should be obvious. The equation for unconditional probabilities in any period t in discrete time is...

$$P[S_t = W_1] = \Big[1 - p\Big]^t$$
$$P[S_t = W_2] = 1 - \Big[1 - p\Big]^t$$

The continuous time equivalents are...

$$P[S_t = W_1] = e^{-\lambda t}$$
$$P[S_t = W_2] = 1 - e^{-\lambda t}$$

where $\lambda = -ln(1 - p)$.

We can also use Linear Algebra to calculate a column vector of probabilities...

$$\Big[P[S_t = W_1], P[S_t = W_2]\Big] = \Big[1.00, 0.00\Big] \times \Big[TransitionMatrix\Big]^t$$

A Hypothetical Case

You have a borrower where you estimate that the probability of that borrower defaulting in any one period is 20%. Once that borrower defaults he stays in default status (i.e. does not have the ability to cure). The borrower is scheduled to make 10 payments of $10,000 over the next 10 periods. The borrower is contractually liable to pay you $100,000 (10 payments of $10,000). Given his default probability, what do you expect to receive?

Period	Prob W1	Prob W2	Payment	Expected
1	0.8000	0.2000	10000	8000
2	0.6400	0.3600	10000	6400
3	0.5120	0.4880	10000	5120
4	0.4096	0.5904	10000	4096
5	0.3277	0.6723	10000	3277
6	0.2621	0.7379	10000	2621
7	0.2097	0.7903	10000	2097
8	0.1678	0.8322	10000	1678
9	0.1342	0.8658	10000	1342
10	0.1074	0.8926	10000	1074

Answer: I only expect to receive the sum of the Expected column, which is $35,705.

If you assume that payments will be received throughout the year then you may be able to use calculus and the continuous time unconditional probability equivalents. If the variable x is the amount that I expect to receive then the answer to our problem is...

$$\begin{aligned}\mathbb{E}\Big[x\Big] &= \int_0^{10} 10000\, e^{-\lambda t} \delta t \\ &= \frac{10000}{-\lambda}\Big[e^{-10\lambda} - 1\Big] \\ &= 40000\end{aligned}$$

where $\lambda = -ln(1 - 0.20) = 0.223$